REGULARIZING A PROBLEM ON THE ENCOUNTER OF MOTIONS IN GAMES THEORY

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The irregular character of the problem [1 to 4] of the minimax of the time-to-encounter of two linear homotypic objects is discussed. The problem is regularized by introducing a discrete scheme of variation of the predicted instant of encounter. A strategy is constructed which guarantees a result which is in a certain sense optimal for the pursuer to within an arbitrarily small $\varepsilon > 0$.

1. Let us supposed that the motions of the pursuing and pursued objects, $y(t)$ and $z(t)$ respectively, are described by the linear differential Eqs.

$$dy/dt = Ay + Bu \tag{1.1}$$

$$dz/dt = Az + Bv \tag{1.2}$$

Here $y(t) = \{y_1(t),\ldots, y_n(t)\}$ and $z(t) = \{z_1(t),\ldots, z_n(t)\}$ are the phase coordinate vectors of the controlled objects; u and v are the r-dimensional vectors of the controlling forces; A and B are constant matrices of the corresponding dimensionalities. Let us isolate certain phase coordinates y_{i_k} and z_{i_k} $(k = 1,\ldots, m \leqslant n)$ whose coincidence at the instant of meeting $t = \vartheta$ is the pursuit goal. Without limiting generality we can assume that $i_k = k$. We shall consider the chosen coordinates as the controlled coordinates [5]. From now on it will be convenient to consider the sets of coordinates $\{y_i\} = y_{[m]}$; $\{z_i\} = z_{[m]}$ $(i = 1,\ldots, m)$ as vectors $\{q_i\} = q$ $(i = 1,\ldots,m)$ in the m-dimensional space Q.

Let us investigate the problem [4] of the minimax of the time-to-encounter of the objects over a portion of the isolated coordinates as a differential positional pursuit game [1 to 3] under the condition that the control resources $u(t)$ and $v(t)$ available for use for $t \geqslant \tau$ are at each given instant τ restricted by integral conditions of the form

$$\int_\tau^\infty \| u(t) \|^2 dt \leqslant \mu^2(\tau), \qquad \int_\tau^\infty \| v(t) \|^2 dt \leqslant \nu^2(\tau) \tag{1.3}$$

From the conditions of the problem the control u at each instant τ must be formed in accordance with the feedback principle on the basis of measurements of the quantities $y(\tau)$ $z(\tau)$, $\mu(\tau)$, and $\nu(\tau)$, i.e. in the form

$$u[\tau] = u[y[\tau],\ z[\tau],\ \mu[\tau],\ \nu[\tau]] \tag{1.4}$$

The pursued player is amenable to both the program control $v = v(t)$, $t \geqslant \tau$, and to the feedback control(*)

*) See [6], p. 535 for an explanation of our use of parentheses and square brackets to denote controls as functions of time.

$$v[\tau] = v[y[\tau],\ z[\tau],\ \mu[\tau],\ \nu[\tau]] \tag{1.5}$$

We must emphasize that the pursuer is not notified of the controls $v = v[t]$ or $v = v(t)$ chosen by the pursued player for instants $t \geqslant \tau$. The strategies u and v, i.e. the sets of functions of the form (1.4) and (1.5), will be considered permissible provided that the following conditions are fulfilled during their realizations $u = u[t]$, $v = v[t]$ or $v = v(t)$: a) limiting conditions (1.3) are not violated; b) Eqs. (1.1) and (1.2) do no lose meaning.

Thus, the problem consists in finding from among the permissible strategies optimal strategies $u^\circ = u^\circ[y, z, \mu, \nu]$ and $v^\circ = v^\circ[y, z, \mu, \nu]$ such that the following condition is fulfilled for all initial values $y(t_0)$, $z(t_0)$, $\mu(t_0)$, and $\nu(t_0)$ (from the specified range of their variation):

$$T_{u^\circ,\, v^\circ} = \min_u \max_v T_{u,\, v} \tag{1.6}$$

where $T_{u,v} = \vartheta_{u,v} - \tau$ is the time-to-encounter of the motions.

The game problem on the encounter of two controlled motions is solved in [5] for $m = n$. In this case problem (1.6) is solved by the external aiming rule which consists in the aiming at each instant $t = \tau$ of the motions $y(t)$ and $z(t)$ towards the point $q^\circ[\tau]$ of osculation of the boundaries of the attainability domains $G_1[\tau, y[\tau], \mu[\tau], \vartheta_0]$ and $G_2[\tau, z[\tau], \nu[\tau], \vartheta_0]$ constructed for the instant of absorption $\vartheta = \vartheta_0$ of the process $z(t)$ by the process $y(t)$ (e.g. see [3], pp. 7 and 8). *If the pursuer constructs his own strategy on the basis of the* extremal aiming rule, then for any permissible v throughout the time-to-encounter the boundaries of the attainability domains $G_1[\tau, y[\tau], \mu[\tau], \vartheta_0]$ and $G_2[\tau, z[\tau], \nu[\tau], \vartheta_0]$ osculate at a single point, and pursuit is successfully accomplished at $t \leqslant \vartheta_0$ provides that the domain G_2 lay inside the domain G_1 at the initial instant of pursuit.

The problem of the minimax of the time-to-encounter of the motions $y_{[m]}(t)$ (1.1) and $z_{[m]}(t)$ (1.2) is more complicated in the case where $m < n$. Here, as with $m = n$, we can construct the extremal strategies

$$u_0[\tau] = \frac{\mu[\tau]}{\mu[\tau] - \nu[\tau]} w^\circ_{x[\tau],\, \varkappa[\tau]}(\tau) \tag{1.7}$$

$$v_0[\tau] = \frac{\nu[\tau]}{\mu[\tau] - \nu[\tau]} w^\circ_{x[\tau],\, \varkappa[\tau]}(\tau) \tag{1.8}$$

where $w^\circ_{x[\tau],\, \varkappa[\tau]}(t)$ $(\tau \leqslant t \leqslant \vartheta^\circ)$ is the optimal program control for the ancillary problem on the transfer of the system

$$\dot{x} = Ax + Bw \tag{1.9}$$

from the state

$$x = x[\tau] = y[\tau] - z[\tau]$$

to the position corresponding to

$$y_{[m]}(\vartheta^\circ) - z_{[m]}(\vartheta^\circ) = x_{[m]}(\vartheta^\circ) = 0$$

under the restriction

$$\left[\int_\tau^\infty \|w(t)\|^2 dt\right]^{1/2} \leqslant \mu[\tau] - \nu[\tau] = \varkappa[\tau] \tag{1.10}$$

and under the condition

$$T_0 = \vartheta^\circ - \tau = \min_w T_w \tag{1.11}$$

This ancillary problem will be called Problem A. *In* addition to Problem A we shall later need ancillary Problem B on the transfer of system (1.9) from the state $x[\tau]$ to the position $x_{[m]}(\vartheta) = 0$ in the specified time $T = \vartheta - \tau$ under the condition

$$\left[\int_\tau^\vartheta \|w(t)\|^2 dt\right]^{1/2} = \min \tag{1.12}$$

Let us introduce the notation

$$\zeta(x[\tau], \tau, \vartheta) = \left[\int_\tau^\vartheta \| w^\circ_{x[\tau],T}(t) \|^2 dt\right]^{1/2} \tag{1.13}$$

where $w^\circ_{x[\tau],T}(t)$ is the solution of problem B. It then turns out that the instant $t = \vartheta^\circ$ of arrival of the motion $x_{[m]}(t)$ at the position $x_{[m]} = 0$ in problem A coincides with the instant of absorption ϑ_0 which is the smallest positive root of Eq.

$$\zeta(x[\tau], \tau, \vartheta) = \mu[\tau] - \nu[\tau] \equiv \varkappa[\tau] \tag{1.14}$$

If the players are guided by the extremal strategies u_0 (1.7) and v_0 (1.8) then the meeting occurs at the instant of absorption $t = \vartheta_0$, just as in the case $m = n$. But now, in contrast to the case $m = n$, the extremal strategy u_0 (1.7) does not guarantee for all permissible v the uniqueness of the point of osculation $q^\circ[\tau]$ throughout the time-to-encounter T.

This statement can be verified, for example, in the case of the motions

$$\frac{dy_1}{dt} = y_2, \qquad \frac{dy_2}{dt} = u_1, \qquad \frac{dy_3}{dt} = y_4, \qquad \frac{dy_4}{dt} = u_2 \tag{1.15}$$

$$\frac{dz_1}{dt} = z_2, \qquad \frac{dz_2}{dt} = v_1, \qquad \frac{dz_3}{dt} = z_4, \qquad \frac{dz_4}{dt} = v_2 \tag{1.16}$$

where it is necessary to effect a meeting only in the coordinates y_1, z_1 and y_3, z_3. The extremal control $u_0[\tau]$ (1.7) in this case is given by

$$u_0 = \left\{-\frac{3}{T_0^2}\frac{\mu}{\mu-\nu}(x_1 + x_2T_0), \; -\frac{3}{T_0^2}\frac{\mu}{\mu-\nu}(x_3 + x_4T_0)\right\} \tag{1.17}$$

where the quantity T_0 is the smallest positive root of Eq. (1.14),

$$\left\{\frac{3\,[(x_1 + x_2T)^2 + (x_3 + x_4T)^2]}{T^3}\right\}^{1/2} = \mu - \nu \tag{1.18}$$

If the pursuer makes use of the control u_0 (1.17) and if the pursued player chooses the control $v(t) = \psi = \text{const}$, where $\|\psi\|$ is a sufficiently small quantity, then there exist initial data $y(t_0)$, $z(t_0)$, $\mu(t_0)$ and $\nu(t_0)$ such that the domains G_1 and G_2 merge at some instant $t = T_*$ prior to the coincidence of $y_1(t)$, $y_3(t)$ and $z_1(t)$, $z_3(t)$. But at the instant of merging of the attainability domains, which in the general case of system (1.1), (1.2) under restrictions (1.3) constitute congruent and similarly oriented ellipsoids, the number of points of osculation q° of their boundaries becomes infinitely large and the extremal aiming rule is violated. At such instants the pursued player has the opportunity of escaping from the pursuer's domain of attainability.

Thus, for $m < n$ the choice of the control $u[\tau]$ on the basis of the extremal aiming rule does not guarantee meeting of the motions $y_{[m]}[t]$ and $z_{[m]}[t]$ in a time $\tau \leqslant t \leqslant \vartheta_0$. It turns out, furthermore [4], that for $m < n$ it is generally impossible to construct a control $u = u^*[y[\tau], z[\tau], \mu[\tau], \nu[\tau]]$ which would ensure meeting within a time $t \leqslant \vartheta_0[\tau]$. These difficulties for the case $m < n$ can be overcome, however, through suitable regularization of the problem.

2. Our earlier paper [4] contains a regularization of the above problem based on the introduction between the boundaries of the domains G_1 and G_2 of the intervening layer afforded by an additional margin $\varepsilon(\tau) > 0$ in the control resource $\mu(\tau)$.

Here we present another regularization of the problem based on a discrete control scheme which allows us to bring the point $y_{[m]}[t]$ into an arbitrarily small neighborhood of the point $z_{[m]}(\vartheta_0)$.

Let us assume that at the initial instant of pursuit $t = t_0$ the domain $G_2[t_0, z(t_0), \nu(t_0), \vartheta_0]$ lay inside the domain $G_1[t_0, y(t_0), \mu(t_0), \vartheta_0]$ and that their boundaries osculated at the single point $q^\circ[t_0]$. The pursuer can then make use of extremal control (1.7), at least until the instant $t = T_*$ at which the domains

$$G_2\ [\tau_*,\ z\,[\tau_*],\ \nu\,[\tau_*],\ \vartheta_0];\ G_1\ [\tau_*,\ y\,[\tau_*],\mu\,\tau_*],\ \vartheta_0]$$

merge. We introduce the notation

$$y\,[\tau_*] = y_*,\ z\,[\tau_*] = z_*,\ \mu\,[\tau_*] = \mu_*,\ \nu\,[\tau_*] = \nu_*$$

The Eq. $\varkappa[\tau_*] = \varkappa_* = \mu_* - \nu_* = 0$ is valid at the instant $t = \tau_*$.

At the instant $t = \tau_*$, i.e. at the point y_*, z_*, $\varkappa_* = 0$, the most natural course is to choose the control $u[\tau_*]$ from the randomization condition [2] for extremal strategies in the case of an infinite number of extremal aiming points. The pursuer can be aimed at any of these points at each instant $t = \tau_*$ with equal probability of success. But by virtue of the symmetry of the attainability domain each extremal control is associated with an extremal control of equal in norm but opposite sign. Hence, the average value of all the extremal strategies at the instant of merging of the attainability domains is equal to zero. Hence, it is most natural (*) to set $u[\tau_*] = u_0[y_*, z_*, \mu_*, \nu_*] = 0$.

However, at subsequent instants $t > \tau_*$ the domains G_1 and G_2 are, as a rule (**), no longer merged, so that it becomes necessary to choose the control $u[y[\tau], z[\tau], \mu[\tau], \nu[\tau]]$ from other considerations, e.g. by once again setting $u = u_0$ (1.7). This renders the right sides of differential Eqs. (1.1) irregular; they turn out to have a discontinuity at the point $x_* = y_* - z_*, \varkappa_* = \mu_* - \nu_* = 0$. It is therefore advisable to convert to a discrete control system. Let us choose a small $\Delta\tau > 0$ and set $u(t) \equiv 0$ for the time $\tau_* \leqslant t < \tau_* + \Delta\tau$. It is easy to show that if the domain G_2 remains inside the domain G_1 throughout the time $t > \tau_*$, then the encounter will occur not later than at the instant $t = \vartheta_0[\tau_*]$. The contrary case is unfavorable to the pursuer.

Let us suppose that the pursued player has chosen a control $v(t)$ $(\tau_* \leqslant t < \tau_* + \Delta\tau)$ such that a portion of the domain

$$G_2\ [\tau_* + \Delta\tau,\ z\,[\tau_* + \Delta\tau],\ \nu\,[\tau_* + \Delta\tau],\ \vartheta_0\,[\tau_*]]$$

lies outside the boundaries of the domain

$$G_1\ [\tau_* + \Delta\tau,\ y\,[\tau_* + \Delta\tau],\ \mu\,[\tau_* + \Delta\tau],\ \vartheta_0\ [\tau_*]]$$

In accordance with Eq. (1.14) a new instant of absorption $\vartheta_0[\tau_* + \Delta\tau]$ occurs at the instant $t = \tau_* + \Delta\tau$, and the domain $G_2[\tau_* + \Delta\tau, z[\tau_* + \Delta\tau], \nu[\tau_* + \Delta\tau], \vartheta_0[\tau_* + \Delta\tau]]$ lies inside the domain $G_1[\tau_* + \Delta\tau, y[\tau_* + \Delta\tau], \mu[\tau_* + \Delta\tau], \vartheta_0[\tau_* + \Delta\tau]]$, touching it at the single point q^*. It is then possible to make use once again of extremal strategy (1.7), aiming towards the point q^* until the domains G_2 and G_1 merge once again.

It is reasonable to hope that by cyclically alternating the extremal aiming rule with a control constructed in a short time $\Delta\tau > 0$ on the basis of the randomization condition we might obtain a regular strategy (an R-strategy) which guarantees encounter at an instant arbitrarily close to the instant of absorption (as $\Delta\tau \to 0$). Unfortunately, this simple technique of choosing the control u does not yield the desired result.

Let us illustrate this for the case of motions (1.15) and (1.16). Let us assume that the instant $t = \tau_* = 0$ when the domains G_1 and G_2 merge has arrived, and that the position $y[\tau_*] = \{-\alpha, 1+\beta, 0, 0\}$, $z[\tau_*] = \{0, \beta, 0, 0\}$ was attained at this instant; in addition, we assume that $\mu[\tau_*] = \nu[\tau_*] = 1$, $\vartheta_0[\tau_*] = \alpha > 0$. Stipulating that $u(t) = \{0, 0\}$, $v(t) = \{\psi, 0\}$ during the time $\Delta\tau$, we can write out Eq. (1.18) for determining the instant of absorption $\vartheta_0[\tau_* + \Delta\tau]$,

$$[1 - \sqrt{1-\psi^2\Delta\tau}]^2(\vartheta - \Delta\tau)^2 - 3\left[\left(\Delta\tau - \psi\frac{\Delta\tau^2}{2} - \alpha\right) + (1 - \psi\Delta\tau)(\vartheta - \Delta\tau)\right]^2 = 0 \quad (2.1)$$

It is easy to see that for a small $\psi > 0$ the smallest positive root of Eq. (2.1) which is

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*) The control $u[\tau_*]$ chosen from the condition $\min_u \max_v d\varepsilon/dt$ is similar. Here ε is an estimate of the possible overhang of the domain G_2 beyond the domain G_1 (in some convenient metric) (see below Section 3)

**) The domains G_1 and G_2 will certainly drift apart provided that the controls u and v do not vanish simultaneously for $t > \tau_*$.

equal to $\vartheta_0[\tau_* + \Delta\tau]$ is arbitrarily large for a sufficiently small $\Delta\tau$. Hence, the control law just described generally cannot insure encounter at instants close to the instant of absorption $\vartheta_0[\tau_*]$. We can also verify that this technique does not guarantee ε-convergence for $t \leqslant \vartheta_0[\tau_*]$.

Let us attempt to construct the R-strategy in a different way. Let us hold the number $\vartheta_0[\tau_*] = \vartheta_*$ fixed and assume once again that for $u(t) \equiv 0$ $(\tau_* \leqslant t < \tau_* + \Delta\tau)$ a portion of the domain $G_2[\tau_* + \Delta\tau, z[\tau_* + \Delta\tau], \nu[\tau_* + \Delta\tau], \vartheta_*]$ has exceeded the boundary of the domain $G_1[\tau_* + \Delta\tau, y[\tau_* + \Delta\tau], \mu[\tau_* + \Delta\tau], \vartheta_*]$. Let $\varepsilon[\tau]$ be the smallest quantity necessary for the domain $G_2[\tau, z[\tau], \nu[\tau], \vartheta_*]$ to lie inside the domain $G_1[\tau, y[\tau], \mu[\tau] + \varepsilon[\tau], \vartheta_*]$ for instants $\tau > \tau_*$. Hence, $\varepsilon[\tau]$ can be determined from the condition

$$\varepsilon[\tau] = \zeta(x[\tau], \tau, \vartheta_*) - \varkappa[\tau] \tag{2.2}$$

where ζ can be computed from Formula (1.13). Once the time $\Delta\tau$ has elapsed we choose the pursuer's extremal control in the form

$$u_\varepsilon[\tau] = \frac{\mu[\tau]}{\varkappa[\tau] + \varepsilon[\tau]} w^\circ_{x[\tau], \zeta[\tau]}(\tau) \tag{2.3}$$

where $w^\circ_{x[\tau], \zeta[\tau]}(t)$ is the solution of problem A under the condition

$$\left[\int_\tau^\infty \|w(t)\|^2 dt\right]^{1/2} \leqslant \varkappa[\tau] + \varepsilon[\tau] = \zeta[\tau] \tag{2.4}$$

Let the pursuer continue to make use of the control $u_\varepsilon[\tau]$ (2.3) until the domains $G_1[\tau, y[\tau], \mu[\tau] + \varepsilon[\tau], \vartheta_*]$ and $G_2[\tau, z[\tau], \nu[\tau], \vartheta_*]$ merge, i.e. until $\zeta[\tau]$ vanishes. After this for the time $\Delta\tau$ we once again set $u(t) \equiv 0$, etc. If the control technique just described did, in fact, guarantee arbitrary smallness of the quantity $\varepsilon(\vartheta_*)$ as $\Delta\tau \to 0$ for all permissible v, then by an instant $t \leqslant \vartheta_*$ the motion $y_{[m]}[t]$ would enter an arbitrarily small neighborhood of the point $z_{[m]}(\vartheta_*)$, which would signify satisfactory solution of the problem. This does not happen, however.

Indeed, from the definitions of the quantities ζ and ε (see (3.13), (3.14), and (3.21) below), we find that for $\zeta = 0$

$$\frac{d\zeta}{d\varepsilon} = \frac{d\zeta / dt}{d\zeta / dt - \|v\|^2 / 2\nu} \tag{2.5}$$

while for small $\zeta > 0$ we have

$$\frac{d\zeta}{d\varepsilon} = 2\,\frac{\varepsilon \|w^\circ\|^2 \zeta^{-2} + (w^\circ / \zeta, \delta v)}{\varepsilon \|w^\circ\|^2 \zeta^{-2} - \|\delta v\|^2 \nu^{-1}} + O(\zeta) \tag{2.6}$$

where $\delta v = v - \nu w^\circ/\zeta$ and where $O(\zeta)$ is an infinitesimal of order ζ. The quantity $d\zeta/dt$ in (2.5) is strictly positive if $v \neq 0$ and vanishes for $v = 0$. If $v = \psi =$ const, where the quantity $\|\psi\|$ is sufficiently small, then $d\zeta/d\varepsilon$ (2.5) is arbitrarily close to unity. However, for $\zeta > 0$ and $\varepsilon < \nu$ we see that, first, $d\zeta/dt < 0$, and second, that the quantity $d\zeta/d\varepsilon$ (2.6) is close to two. Hence, for such a v (however small our $\Delta\tau$) the function $\varepsilon(t)$ can increase proportionally to time with a proportionality coefficient which does not tend to zero as $\Delta\tau \to 0$. This means that $\varepsilon(t)$ cannot be made smaller than a preselected positive number by the instant $t = \vartheta_*$. Hence, as $\Delta\tau \to 0$ this method of constructing the control u for $v = \psi =$ const gives rise to a characteristic slippage state which produces a considerable increase in $\varepsilon(t)$.

3. In this section we shall develop a solution of the problem which will enable us to overcome the difficulties described above. The modification of the problem about to be discussed is based on a discrete scheme of variation of the control u which is accompanied, as towards the end of Section 2, by braking of the quantity $\vartheta_0[\tau]$. However, we shall now make use of a smoothed extremal control. This will enable us to circumvent the difficulties which we confronted in Section 2. Let us now describe the proposed method of constructing the control.

Let $\{\tau_k\}$ $(k = 0, 1, \ldots)$ be a sequence of instants of time; let $\tau_{k+1} - \tau_k = \Delta$. Let the symbol $u_\Delta[t]$ denote the control u, which changes only at the instants $t = \tau_k$. The value of $u_\Delta[t]$ in the interval $[\tau_k, \tau_{k+1})$ is then determined only by the quantities realized by the instant $t = \tau_k$. In choosing the control u in this way we take the quantity

$$\gamma_u = \sup_\sigma \{\limsup_{\Delta\to 0} [\sup_v T^\sigma_{u_\Delta, v}]\} \ (\sigma > 0) \tag{3.1}$$

as our criterion of the pursuit results.

Here the number $T^\sigma_{u_\Delta v}$ denotes the instant $t = \tau + T^\sigma_{u_\Delta, v}$ at which the condition

$$\| y_{[m]}[\tau + T] - z_{[m]}[\tau + T] \| \leqslant \sigma \tag{3.2}$$

is fulfilled for the first time for the chosen control $v(t)$ and the chosen law for constructing the control $u_\Delta^\cdot[t]$, and for the stipulated initial state $y[\tau]$, $z[\tau]$, $\mu[\tau]$, $\nu[\tau]$. The problem now consists in choosing the optimal control $u^\circ[t]$ which gives the minimum

$$T^\circ = \gamma_{u^\circ_\Delta} = \min_u \gamma_u \tag{3.3}$$

for any initial conditions $y[\tau]$, $z[\tau]$, $\mu[\tau]$, $\nu[\tau]$ from the domain of their possible variation. As our arguments which determine the extremal control $u_\Delta[t]^\circ$ in the intervals $[\tau_k, \tau_{k+1})$ we take the values of the variables $y[\tau_k]$, $z[\tau_k]$, $\mu[\tau_k]$, $\nu[\tau_k]$ and of the ancillary variable $\vartheta[\tau_k]$ whose meaning will be explained below. Thus, we construct the control $u_\Delta[t]^\circ$ in the form

$$u_\Delta[t]^\circ = u[y[\tau_k], z[\tau_k], \mu[\tau_k], \nu[\tau_k], \vartheta[\tau_{k-1}]]^\circ \ (\tau_k \leqslant t < \tau_{k+1}) \tag{3.4}$$

The algorithm which determines the right side of (3.4) and the sequence of values $\vartheta[\tau_k]$ is as follows. Let pursuit begin at $t = \tau_0$. From now on we shall always assume that the inequality $\mu[\tau_0] > \nu[\tau_0]$ is fulfilled at $t = \tau_0$ and that Problem A has the finite solution $T_0[\tau_0]$ for $t = \tau_0$, $x[\tau_0] = y[\tau_0] - z[\tau_0]$, $\varkappa[\tau_0] = \mu[\tau_0] - \nu[\tau_0]$.

Let us set $\vartheta[\tau_{-1}] = \vartheta[\tau_0] = \tau_0 + T_0[\tau_0]$. The control $u_\Delta[t]^\circ$ in Formula (3.4) is defined by the following two equations of differing form:

if $\varkappa[\tau_0] > 0$

$$u_\Delta[t]^\circ = \frac{\mu[\tau_0]}{\varkappa[\tau_0]} w^\bullet_{x[\tau_0], \varkappa[\tau_0]}(\tau_0) \qquad (\tau_0 \leqslant t < \tau_1) \tag{3.5}$$

if $\varkappa[\tau_0] = 0$

$$u_\Delta[t]^\circ \equiv 0 \qquad (\tau_0 \leqslant t < \tau_1) \tag{3.6}$$

Now let $\tau = \tau_k > \tau_0$. We shall determine the quantities $\vartheta[\tau_j]$ recurrently; thus, we assume that $\vartheta[\tau_{k-1}]$ is known at the instant $\tau = \tau_k$. If the quantity $\varkappa[\tau_k] = \mu[\tau_k] - \nu[\tau_k] > 0$ was realized at the instant $\tau = \tau_k$, then we once again solve problem A for the realized $\tau = \tau_k$, $x[\tau_k] = y[\tau_k] - z[\tau_k]$, $\varkappa = \varkappa[\tau_k]$.

Let us assume first that solution of this problem yields

$$T_0[\tau_k] \leqslant \vartheta[\tau_{k-1}] - \tau_k \tag{3.7}$$

We can then set $\vartheta[\tau_k] = \tau_k + T_0[\tau_k]$ and

if $\varkappa[\tau_k] > 0$

$$u_\Delta[t]^\circ = \frac{\mu[\tau_k]}{\varkappa[\tau_k]} w^\bullet_{x[\tau_k], \varkappa[\tau_k]}(\tau_k) \qquad (\tau_k \leqslant t < \tau_{k+1}) \tag{3.8}$$

if $\varkappa[\tau_k] = 0$

$$u_\Delta[t]^\circ \equiv 0 \qquad (\tau_k < t < \tau_{k+1}) \tag{3.9}$$

On the other hand, if problem A under consideration does not have a solution which satisfies condition (3.7), or if the realized quantity $\varkappa[\tau_k] < 0$ is smaller than zero, then what we must do is solve problem B under the conditions $\tau = \tau_k$, $x[\tau_k]$, and $T = \vartheta[\tau_{k-1}] - \tau_k$. Let the solution of this problem yield the quantity $\zeta[\tau_k]$. It is clear that now $\zeta[\tau_k] > \varkappa[\tau_k]$. The next step is to solve problem A for $\tau = \tau_k$, $x[\tau_k]$ and under the condition

$$\left[\int\limits_{\tau_k}^{\infty} \|w(t)\|^2 d\tau\right]^{1/2} \leqslant \zeta[\tau_k]$$

This solution clearly gives us the quantity $T_0[\tau_k] \leqslant \vartheta[\tau_{k-1}] - \tau_k$. Let us set $\vartheta[\tau_k] = \tau_k + T_0[\tau_k]$. The value of the control $u_\Delta[t]^\circ$ now depends on the value of

$$\varepsilon[\tau_k] = \zeta[\tau_k] - \varkappa[\tau_k] > 0 \tag{3.10}$$

Specifically, we set

if $\zeta[\tau_k] < \varepsilon[\tau_k]$

$$u_\Delta[t]^\circ = \frac{\mu[\tau_k]}{\varepsilon[\tau_k]} w^*_{x[\tau_k],\zeta[\tau_k]}(\tau_k) \tag{3.11}$$

if $\zeta[\tau_k] > \varepsilon[\tau_k]$

$$u_\Delta[t]^\circ = \frac{\mu[\tau_k]}{\zeta[\tau_k]} w^\circ_{x[\tau_k],\zeta[\tau_k]}(\tau_k) \tag{3.12}$$

Construction is carried on until $\vartheta[\tau_k] > \tau_k$. Control (3.4) constructed in this way solves problem (3.1) to (3.3). It turns out here that $T^\circ[\tau] = T_0[\tau]$. Let us prove this result. First, let $\tau = \tau_0$. To begin with, let us verify that for any permissible choice of $v[t]$ $(t > \tau_0)$ and for $u = u_\Delta[t]^\circ$ the required σ-convergence (3.2) of the motions $y_{[m]}[t]$ and $z_{[m]}[t]$ occurs not later than at the instant $t = \tau_0 + T_0[\tau_0]$ provided the quantity Δ is sufficiently small.

To show this it is sufficient to verify that for small Δ the quantity $\varepsilon[\tau_k]$ remains arbitrarily small with increasing time τ_k provided that $\vartheta[\tau_k] - \tau_k > \eta(\varepsilon)$. (Here $\eta(\varepsilon)$ and ε are infinitesimals). In fact, as already noted in Section 2, the quantity $\varepsilon[\tau_k]$ is the increment which must be added to the control resource $\mu[\tau_k]$ in order for the attainability domain $G_1[\tau_k, y[\tau_k], \mu[\tau_k] + \varepsilon[\tau_k], \vartheta[\tau_k]]$ to encompass the domain $G_2[\tau_k, z[\tau_k], \nu[\tau_k], \vartheta[\tau_k]]$.

But if the quantity $\varepsilon[\tau_k]$ is small, then the domain $G_2[\tau_k, z[\tau_k], \nu[\tau_k], \vartheta[\tau_k]]$ lies in a small σ-neighborhood of the domain $G_1[\tau_k, y[\tau_k], \mu[\tau_k], \vartheta[\tau_k]]$. Since (by construction) $\vartheta[\tau_k] \leqslant \tau_0 + T_0[\tau_0]$ and since the domains G_1 and G_2 contract to a point as $\tau_k \to \vartheta[\tau_k]$, we see that a sufficiently small $\varepsilon[\tau_k]$ does, in fact, guarantee the required σ-convergence of the motions $y_{[m]}[t]$ and $z_{[m]}[t]$ for all $\tau_0 \leqslant \tau_k \leqslant \vartheta[\tau_k] - \eta(\varepsilon)$.

From now on it will be convenient to represent the variation of the system parameters with time t on the plane $\{\varepsilon\, \zeta\}$. To prove the above statement it is enough to show that for any $\varepsilon^* > 0$ and $\eta^* > 0$ we can choose a number $\Delta^\circ > 0$ such that for $\Delta < \Delta^\circ$ the control $u_\Delta[t]^\circ$ keeps the motion $\{\varepsilon[t], \zeta[t]\}$ in the domain H, i.e. that $\{\varepsilon[t] < \varepsilon^*, \zeta[t] > 0\}$ for $\vartheta[\tau_k] - \tau_{k+1} > \eta^*$.

Let us consider the domain $\varepsilon > \varepsilon^\circ$, $\zeta > 0$ (see Fig. 1), where ε° is a sufficiently small number smaller than ε^*. Let us show that in this domain the quantity $\varepsilon[t]$ can-

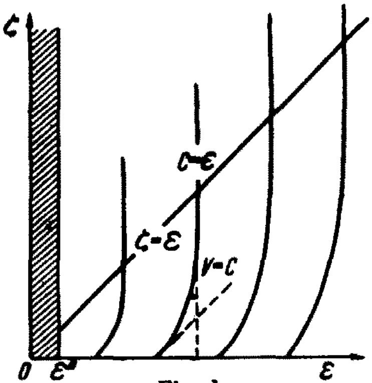

Fig. 1

not increase too rapidly. We assume first that in the domain H the control $u_\Delta\ [t]$ is formed in each interval $[\tau_k,\ \tau_{k+1})$ not in accordance with the above rule, but rather on the basis on Formulas similar to (3.11) and (3.12) in whose right sides the argument τ_k has been replaced by t. (Here the quantity $\vartheta\ [\tau_k]$ which is involved in the definition of the quantity $\zeta[\tau]$ remains constant for $\tau = t$ throughout the interval $[\tau_k,\ \tau_{k+1})$).

The following equations are valid in the domain $\varepsilon > 0,\ \zeta > 0$:

$$\frac{d\zeta}{dt} = -\frac{1}{2\zeta}[\|w^\circ\|^2 + 2(w^\circ,\ \delta w)] \qquad (3.13)$$

$$\frac{d\varepsilon}{dt} = \frac{\varepsilon}{2\zeta^2}\|w^\circ\|^2 + \frac{\|\delta u\|^2}{2\mu} - \frac{\|\delta v\|^2}{2\nu} \qquad (3.14)$$

These equations follow directly from the definitions of the quantities ζ and ε. Here $w^\circ[\tau] = w^\circ_{x[\tau],T}(\tau)$ is the solution of Problem B for $x[\tau]$, $T = \vartheta[\tau_k] - \tau$; $\delta w = \delta u - \delta v - \varepsilon w^\circ / \zeta$, $\delta u = u_\Delta\ [t] - \mu\ [t]\ w^\circ\ [t] / \zeta\ [t]$; $\delta v = v\ [t] - \nu w^\circ\ [t] / \zeta\ [t]$. In the domain $\zeta \geqslant \varepsilon \geqslant \varepsilon_0$ we have $\delta u = 0$, so that

$$\frac{d\varepsilon}{dt} = \frac{\varepsilon}{2\zeta^2}\|w^\circ\|^2 - \frac{\|\delta v\|^2}{2\nu} \leqslant \lambda\varepsilon \qquad (\lambda = \text{const} > 0) \qquad (3.15)$$

since [7] for $\vartheta\ [\tau_k] - \tau \geqslant \eta^*$ the quantity $\|w^\circ\|/\zeta$ is uniformly bounded. Integrating inequality (3.15), we find that throughout the time $t_\alpha \leqslant t < t_\beta$ during which the trajectory $\{\varepsilon\ [t],\ \zeta\ [t]\}$ remains in the domain $\varepsilon > 0,\ \zeta \geqslant \varepsilon$, we have the inequality

$$\varepsilon\ [t] \leqslant \varepsilon\ [t_\alpha]\, e^{\lambda\, [t - t_\alpha]} \qquad (3.16)$$

Let us consider the function

$$V\ (\varepsilon,\ \zeta) = (\varepsilon^2 - \varepsilon\zeta + \zeta^2)^{1/2}\exp\left\{\frac{1}{\sqrt{3}}\left[-\frac{\pi}{6} + \operatorname{arctg}\frac{2\varepsilon - \zeta}{\sqrt{3}\zeta}\right]\right\} \qquad (3.17)$$

in the domain $0 \leqslant \zeta < \varepsilon$, the datum levels $V(\varepsilon,\ \zeta) = C > 0 =$ const appear in the Fig. 1. The total derivative dV/dt of this function for $\varepsilon = \varepsilon\ [t],\ \zeta = \zeta\ [t] > 0$ in the case $u = u_\Delta\ [t]$ is given by Expression

$$\frac{dV}{dt} = \frac{\varepsilon V}{\varepsilon^2 - \varepsilon\zeta + \zeta^2}\left\{\frac{\|w^\circ\|^2}{2\zeta^2}\left[\varepsilon - \zeta\,\frac{(\zeta - \varepsilon)}{\varepsilon} + 2\,(\zeta - \varepsilon) - \frac{\mu\,(\zeta - \varepsilon)^2}{\varepsilon^2}\right] - \right.$$

$$\left. - \frac{\|\delta v\|^2}{2\nu} + \frac{\zeta - \varepsilon}{\varepsilon\zeta}(w^\circ,\ \delta v)\right\} \qquad (3.18)$$

and admits of the estimate

$$dV/dt \leqslant \lambda V \qquad (3.19)$$

This estimate implies that throughout the time $t_\alpha \leqslant t \leqslant t_\beta$ during which the trajectory $\{\varepsilon\ [t],\ \zeta\ [t]\}$ remains in the domain $\varepsilon > 0,\ 0 < \zeta \leqslant \varepsilon$ we have the inequality

$$V\ [t] \leqslant V\ [t_\alpha]\, e^{\lambda\,(t - t_\alpha)} \qquad (3.20)$$

Since $\zeta \leqslant \varepsilon$ for $V \geqslant \varepsilon$ and $V = \varepsilon$ for $\zeta = \varepsilon$, the estimates (3.16) and (3.20) imply that Inequality (3.16) applies throughout the time when $t \geqslant t_\alpha$ during which the trajectory $\{\varepsilon[t],\ \zeta\ [t]\}$ remains in the domain $\varepsilon > 0,\ \zeta > 0$

Now let us consider the variation of V for $u = u_\Delta\ [t]$ once the trajectory $\{\varepsilon\ [t],\ \zeta\ [t]\}$ has emerged onto the boundary $\zeta = 0$ of the domain $\varepsilon > 0,\ \zeta \geqslant 0$. In this case the derivative $d\zeta/dt$ is given by Eq.

$$\left(\frac{d\zeta}{dt}\right)_{+0} = \lim_{\Delta t \to +0}\frac{\zeta - 0}{\Delta t} = \sqrt{(D^{-1}H^{[m]}v,\ H^{[m]}v)} \qquad (3.21)$$

where D and $H^{[m]}$ are certain matrices which can be computed in the usual way.

Furthermore,

$$\frac{d\varepsilon}{dt} = \frac{d\zeta}{dt} - \frac{\|v\|^2}{2\nu} \tag{3.22}$$

so that for the derivative dV/dt we obtain

$$\frac{dV}{dt} = \frac{V}{\varepsilon}\left[\frac{d\varepsilon}{dt} - \frac{d\zeta}{dt}\right] = -\frac{V}{\varepsilon}\frac{\|v\|^2}{2\nu} \tag{3.23}$$

i.e. the function V does not increase for $\varepsilon > 0$ and $\zeta = 0$ with the control $u = u_\Delta[t]$ estimate (3.16) is therefore fulfilled for all times in the domain $\varepsilon > 0$, $\zeta \geqslant 0$. Hence, if the quantity $\varepsilon[t_\alpha]$ is sufficiently small, then subsequently at all times $t \leqslant \vartheta[\tau_k] - \eta^* \leqslant \vartheta[\tau_0] - \eta^*$ the value of $\varepsilon[t]$ will remain sufficiently small provided that $u = u_\Delta[t]$.

It now remains for us to estimate the effect of converting from the control $u_\Delta[t]$ to the control $u_\Delta[t]^\circ$ under investigation here. Without presenting detailed estimates for Eqs. (1.1), (3.13), (3.14), (3.21) and (3.22), which can be obtained without much difficulty in the domain $\varepsilon > \varepsilon^\circ$ by the usual procedure for converting from a differential to a finite difference scheme of variable substitution, we shall merely state the final result: the deviation of the trajectories under investigation for $u = u_\Delta[t]$ per step $[\tau_k, \tau_{k+1})$ is on the order of $o(\Delta)$, where $o(\Delta)$ is an infinitesimal of order higher than Δ (for the entire time during which the representing point $\{\varepsilon, \zeta\}$ remains in the domain $\varepsilon^\circ > \varepsilon^\circ$); this estimate $o(\Delta)$ is uniform for each fixed $\varepsilon^\circ > o$. Estimate (3.16) then implies the required smallness of the quantity $\varepsilon[\tau_k]$, since $\varepsilon[\tau_0] = 0$.

Thus, for $\tau = \tau_0$ the control $u = u_\Delta[t]^\circ$ does, in fact, insure σ-convergence of the motions $y_{[m]}[t]$ and $z_{[m]}[t]$ for $t < \tau_0 + T_0[\tau_0]$, provided the quantity $\Delta > 0$ is sufficiently small. Similarly, since $\varepsilon[\tau_{k^*}]$ is small for small Δ for any intermediate value $\tau = \tau_{k^*}$, $k^* > 0$ (by virtue of what was proved above), we see that the control $u_\Delta[t]^\circ$ for $t \geqslant \tau_{k^*}$ guarantees σ-convergence of the motions $y_{[m]}[t]$ and $z_{[m]}[t]$ for $t < \tau_{k^*} + T_0[\tau_{k^*}]$.

On the other hand, taking $v^*[t] = \mu[t]\, u^*[t] / \nu[t]$ we see that for any $u^*[t]$ for $t > \tau_{k^*}$ arbitrary σ-convergence cannot be guaranteed for $t < \tau_{k^*} + T_0[\tau_{k^*}] - \alpha$, where $\alpha = \text{const} > 0$. This implies the optimality of the control $u_\Delta[t]^\circ$ for problem (3.1) to (3.3).

4. The difficulties considered in the present paper are immediately removed if we assume [1, 8 and 9] the possibility of constructing the control $u[t]$ in the form

$$u[t] = u[y[t],\ z[t],\ \mu[t],\ \nu[t],\ v[t]] \tag{4.1}$$

since here in the critical situation where $\mu = \nu$ it is sufficient, for example, to set $u[t] = v[t]$. If such direct discrimination of the motion $z[t]$ is undesirable, then the quantity $v[t]$ can be allowed for indirectly in computing the control u. This can be done by again introducing some aftereffect in the control law u. Taking γ_u (3.1) as our criterion of the pursuit result, we construct Eq.

$$u_\Delta[t] = u_\Delta(t,\ y[\tau_k],\ z[\tau_k],\ \mu[\tau_k],\ \nu[\tau_k],\ z[\tau_{k-1}],\ \nu[\tau_{k-1}])_0 \qquad (\tau_k \leqslant t < \tau_{k+1}) \tag{4.2}$$

in the following way. From the values of $y[\tau_k]$, $\mu[\tau_k]$, $z[\tau_{k-1}]$ and $\nu[\tau_{k-1}]$ we determine the instant of absorption $\vartheta^\circ[\tau_k, \tau_{k-1}]$ when the attainability domain $G_2[\tau_{k-1}, z[\tau_{k-1}], \nu[\tau_{k-1}], \vartheta - \Delta]$ first lies inside the domain $G_1[\tau_k, y[\tau_k], \mu[\tau_k], \vartheta]$. We denote by $\varepsilon[\tau, \tau_*; \vartheta]$ the quantity

$$\varepsilon[\tau, \tau_*; \vartheta] = \max_z \min_y \|y - z\| \tag{4.3}$$

for y from $G_1[\tau, y[\tau], \mu[\tau], \vartheta]$ and z from $G_2[\tau_*, z[\tau_*], \nu[\tau_*], \vartheta - \Delta]$. Clearly, from the chosen $\vartheta^\circ[\tau_k, \tau_{k-1}]$ we have $\varepsilon[\tau_k, \tau_{k-1}, \vartheta^\circ[\tau_k, \tau_{k-1}]] = 0$.

We now define the control $u_\Delta[t]$ (4.2) on the interval $[\tau_k, \tau_{k+1}]$ as the optimal program control which guarantees fulfillment of the condition

$$\varepsilon[\tau_{k+1}, \tau_k; \vartheta^\circ[\tau_k, \tau_{k-1}]] = 0 \tag{4.4}$$

i.e. keeps the domain

$$G_2[\tau_k, z[\tau_k], \nu[\tau_k], \vartheta^\circ[\tau_k, \tau_{k-1}] - \Delta]$$

inside the domain

$$G_1[\tau_{k+1}, y[\tau_{k+1}], \mu[\tau_{k+1}], \vartheta^\circ[\tau_k, \tau_{k-1}]]$$

and maximizes the distance between the boundaries of these domains in some convenient metric. Clearly, it is always possible to guarantee fulfillment of Eq. (4.4) in our case. In fact, this condition already guarantees, for example, an extremal control $u^*(t) = u_0[t]$ aimed at the point of osculation of the boundaries of the domains

$$G_1[t, y[t], \mu[t], \vartheta^\circ[\tau_k, \tau_{k-1}]]$$

$$G_2[t - \Delta, z(t - \Delta), \nu[t - \Delta] + \beta(t), \vartheta^\circ[\tau_k, \tau_{k-1}]] \quad (\beta(t) \geqslant 0)$$

as long as

$$\mu[t] > \nu[t - \Delta] + \beta(t) \quad \text{or} \quad u^*(t) = v(t - \Delta),$$

where $\mu[t] = \nu[t - \Delta] + \beta(t)$. But Eq. (4.4) guarantees fulfillment of the inequality $\vartheta^\circ[\tau_{k+1}, \tau_k] \leqslant \vartheta^\circ[\tau_k, \tau_{k-1}]$ so that we can continue our construction of the control $u_\Delta[t]$ in the same way for $\tau_{k+1} \leqslant t < \tau_{k+2}$ etc. The method of constructing the control $u_\Delta[t]$ just described insures convergence of the point $y_{[m]}[t]$ and $z_{[m]}[t-\Delta]$ at the instant $t \leqslant \vartheta^\circ[\tau_{-1}, \tau_0]$, so that for a sufficiently small $\Delta > 0$ it also insures the required convergence of the points $y_{[m]}[t]$ and $z_{[m]}[t]$.

This regularization applies not only to linear homotypic objects, but also in the very general case where successful completion of the pursuit process is possible with the control $u[t]$ taken in the form (4.1) (although the problem of the minimax character of the control u generally does not require investigation). The conditions which guarantee encounter by means of control (4.1) are known for a broad class of problems with restrictions imposed on the instantaneous values of the controlling forces (e.g. see [1, 8 and 9]). Such conditions can also be derived for problems with integral restrictions on the controls. We note, in particular, that fulfillment of the relation

$$\max_v \min_u \left\{ \frac{\partial}{\partial t} \varepsilon[t + \tau, t + \tau; \vartheta^*[\tau]] \right\}_{+0}^{+\infty} = 0 \tag{4.5}$$

at each instant $t = \tau$ of pursuit at all points of the phase space where the realizations of the quantities $\{y[\tau], z[\tau], \mu[\tau], \nu[\tau]\}$ might occur is clearly sufficient for the successful completion of pursuit under the condition (4.4).

From Eqs. (4.3) and (4.4) we infer immediately that this condition is fulfilled in a self-evident way in our case. In the general case more or less effective sufficient conditions which insure convergence of the motions $y_{[m]}[t]$ and $z_{[m]}[t]$ and are implied by the conditions (4.5) can be constructed in terms of the planes tangent to the attainability domains $G_1[\tau, y[\tau], \mu[\tau], \vartheta^\circ[\tau]]$ and $G_2[\tau, z[\tau], \nu[\tau], \vartheta^\circ[\tau]]$; this reduces to the investigation of functions similar to the quantity ε (4.3). The regularization described in the present Section requires theoretical and experimental study of the stability of the corresponding computational scheme. Such a study is important in view of the involvement in the scheme of the close quantities $z[\tau_k]$, $z[\tau_{k-1}]$ and $\nu[\tau_k]\,\nu[\tau_{k-1}]$. It appears that so far there has been no sufficiently complete inquiry into this problem.

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